2 THE UNIVERSAL QUANTIFIER

The last section affords a preliminary sketch of a new formal language; we can now turn to the matter of testing arguments expressed in it. Since part of our language is just the propositional calculus itself, we take over into the predicate calculus the propositional connectives and propositional variables if we need them—all our earlier rules continue to be of service under the understanding that they are extended to the new symbolism. But we need additional rules for the handling of quantifiers in argument: four such, in fact—an introduction and an elimination rule for the universal and for the existential quantifier. We consider the universal quantifier first.

The elimination rule for the universal quantifier is concerned with the use of a universal proposition as a *premiss* to establish some conclusion, whilst the introduction rule is concerned with what is required by way of premiss for a universal proposition as *conclusion*. It is helpful to bear in mind the corresponding rules for ' & ', for there is a close similarity between ' & ' and the universal quantifier, as the following remarks suggest.

In particular arguments involving quantifiers, we usually have a particular group of objects, called our *universe of discourse*, in mind. For example, in algebra the variables ' *x* ', ' *y* ', ' *z* ', . . . are understood to range over numbers, so that our universe of discourse here is the set of all numbers; and, in discussing (28)–(33) in the last section, we explicitly restricted our universe of discourse to the set of all people. Our universe of discourse, in fact, is generally the understood range of our variables ' *x* ', ' *y* ', ' *z* ', ...

By way of illustration, let us suppose that our universe of discourse contains exactly 3 objects (what they are will not matter) whose proper names are ' *m* ', ' *n* ', and ' *o* '. Then to affirm that everything has property *F* will, for this universe, be to affirm that *m* has *F* and *n* has *F* and *o* has *F*. Thus

(1) $(x)Fx$

is intuitively equivalent, in this universe, to the complex conjunction with 3 conjuncts

(2) $Fm \; \& \; Fn \; \& \; Fo$.

Now by an obvious extension of &E, we could naturally derive as conclusion from (2) any one of the conjuncts separately, Fm, Fn, Fo. Analogously, our rule of universal quantifier elimination (UE) will allow us to infer that *any particular object* has F from the premiss that *all* things have F. The rule can be seen as a natural extension of &E, when we realize that affirming a proposition such as $(x)Fx$ is generally a condensed way of affirming a complex conjunction.

In fact, if all objects in a given universe had names which we knew and there were only finitely many of them, then we could always replace a universal proposition about that universe by such a complex conjunction. It is because these two requirements are not always met that we need universal quantifiers. For example, we may wish to say that all natural numbers [1] have a certain property F; this amounts to saying that 0 has F, and 1 has F, and 2 has F, and so on; but, there being infinitely many numbers, we are barred from actually completing the desired conjunction, and we fall back on the quantifier to do the job. Because our universe of discourse may be infinite in size, we cannot say that a universal proposition is *equivalent* to a complex conjunction, but it is true that the analogy with '&' is intuitively very helpful.

Hence the justification for UE is that, if everything has a certain property, any particular thing must have it, and UE will enable us to pass from $(x)Fx$ to conclusions such as Fm and Fn, and from $(x)(Fx \rightarrow Gx)$ to $Fm \rightarrow Gm$ and $Fn \rightarrow Gn$ (if everything is such that it has G if it has F, then in particular m has G if m has F, n has G if n has F). The rule is exemplified in the proof of the following elementary sequent:

100 $Fm, (x)(Fx \rightarrow Gx) \vdash Gm$

1	(1) Fm	A
2	(2) $(x)(Fx \rightarrow Gx)$	A
2	(3) $Fm \rightarrow Gm$	2 UE
1,2	(4) Gm	1,3 MPP

[1] By the *natural numbers* are meant the numbers 0, 1, 2, 3, etc. They are sometimes called also the *non-negative integers*, the *positive integers* being the numbers 1, 2, 3, etc.

100 exhibits the form of such obviously sound arguments as the logically famous

> (3) Socrates is a man; all men are mortal; therefore Socrates is mortal

(letting ' *m* ' be Socrates, ' *F* ' be being a man, and ' *G* ' be being mortal). We are now also in a position to validate the Tweety and oxygen arguments ((4) and (5) of the last section). Their common form (compare (7) of the last section) is proved as the following sequent:

101 $Fm, (x)(Fx \rightarrow -Gx) \vdash -Gm$

1	(1) Fm	A
2	(2) $(x)(Fx \rightarrow -Gx)$	A
2	(3) $Fm \rightarrow -Gm$	2 UE
1,2	(4) $-Gm$	1,3 MPP

The application of UE at line (3) is exactly like its application at the same line in the proof of 100: if everything with *F* lacks *G*, then in particular if *m* has *F m* lacks *G*.

The rule of universal quantifier introduction (UI) is designed for establishing as conclusions universal propositions. By the analogy with ' & ', to establish, say for our earlier universe of 3 objects, that everything has *F*, we should establish first that *m* has *F*, that *n* has *F*, and that *o* has *F*. Then, by an obvious extension of &I, we are sure that everything has *F*. This technique will be of no avail, however, if our universe is infinitely large or if we do not have names for all objects in the universe. We evidently require a new device.

Think of what Euclid does when he wishes to prove that all triangles have a certain property; he begins ' let ABC be a triangle ', and proves that ABC has the property in question; then he concludes that *all* triangles have the property.[1] What here is ' ABC '? Certainly not the *proper name* of any triangle, for in that case the conclusion would not follow. For example, given that Khrushchev is bald, it does not follow that everyone is bald. It is natural to view ' ABC ' as the name of an *arbitrarily selected triangle*, a particular triangle certainly but any one you care to pick. For if we can show

[1] See, for example, Euclid, *The Elements*, I. Propositions 16-21.

that an arbitrarily selected triangle has *F*, then we can soundly draw the conclusion that all triangles have *F*.

We introduce, therefore, the letters 'a', 'b', 'c', . . . to be names (*not* proper names) of arbitrarily selected objects in the universe of discourse, and call them for short *arbitrary names*. Then, with important reservations to be made later, if we can show that Fa (an arbitrarily selected object has *F*) then we can conclude that $(x)Fx$.

In effect, a proof of Fa is tantamount to a proof of all the required conjuncts in the desired 'conjunction' $(x)Fx$. In the 3-object universe above, to prove Fa is to prove Fm, Fn, and Fo. For we can take m as the arbitrarily selected a, and n, and o. In the case where the universe is infinitely large, proving Fa is tantamount to proving infinitely many conjuncts, for we can select as a any object in the universe.

Hence the justification for UI is that, with certain reservations, if an arbitrarily selected object can be shown to have a property, everything must have it, and UI will enable us to pass from premisses such as Fa or Fb to conclusion $(x)Fx$, and from $Fa \rightarrow Ga$ or $Fb \rightarrow Gb$ to $(x)(Fx \rightarrow Gx)$ (if an arbitrarily selected object has *G* if it has *F*, then everything with *F* has *G*). With the adoption of new letters 'a', 'b', 'c' goes a natural extension of UE: from $(x)(Fx \rightarrow Gx)$, for example, we can conclude not only that $Fm \rightarrow Gm$ but also that $Fa \rightarrow Ga$, $Fb \rightarrow Gb$, and so on (arbitrarily selected objects from the universe are after all particular objects in the universe, so that what holds of everything holds of them too). The rule UI and this extension of UE are both illustrated in the following proofs:

102 $(x)(Fx \rightarrow Gx), (x)(Gx \rightarrow Hx) \vdash (x)(Fx \rightarrow Hx)$

1	(1)	$(x)(Fx \rightarrow Gx)$	A
2	(2)	$(x)(Gx \rightarrow Hx)$	A
1	(3)	$Fa \rightarrow Ga$	1 UE
2	(4)	$Ga \rightarrow Ha$	2 UE
1,2	(5)	$Fa \rightarrow Ha$	3,4 SI(S) 1.2.1(*i*)
1,2	(6)	$(x)(Fx \rightarrow Hx)$	5 UI

To prove that $(x)(Fx \rightarrow Hx)$, we aim to prove $Fa \rightarrow Ha$ (to prove that everything with *F* has *H* we aim to prove that an arbitrarily selected object with *F* has *H*). From assumptions (1) and (2) by UE in its newly extended form we have (3) $Fa \rightarrow Ga$ and (4) $Ga \rightarrow Ha$; the desired $Fa \rightarrow Ga$ now follows by propositional calculus reasoning, steps embodied in a sequent from Chapter 1, which we here abbreviate by SI. (Strictly, we have not proved that SI is obtainable as a derived rule for the predicate calculus, but the extension of our demonstration in Chapter 2, Section 2, to the new formal language is in fact immediate.) This proof is typical of predicate calculus work where both assumptions and conclusions are universally quantified: we drop the universal quantifiers from assumptions, changing variables to arbitrary names, apply *propositional calculus* steps, and finally reintroduce a universal quantifier by UI. Here is another example.

103 $(x)(Fx \rightarrow Gx), (x)Fx \vdash (x)Gx$

1	(1) $(x)(Fx \rightarrow Gx)$	A
2	(2) $(x)Fx$	A
1	(3) $Fa \rightarrow Ga$	1 UE
2	(4) Fa	2 UE
1,2	(5) Ga	3,4 MPP
1,2	(6) $(x)Gx$	5 UI

To prove $(x)Gx$ by UI we aim for Ga, which follows by MPP from $Fa \rightarrow Ga$ and Fa, obtainable from the assumptions by UE.

As already indicated, some restriction has to be placed on the free use of UI, if fallacies are to be avoided. The following illustration should help to show why. Suppose that, in a geometrical context, we arbitrarily select a shape a and assume (i) that it is acute-angled (that is, that none of its angles are as great as a right angle), and (ii) that it is rectilinear (that is, that it is formed by straight lines); then by elementary geometrical reasoning we can conclude that a is a triangle. Expressing (i) by 'Aa', (ii) by 'Ra', and the conclusion by 'Ta', we have that Ta follows from Aa and Ra. Hence, by a step of CP, given that Aa, $Ra \rightarrow Ta$. If we now apply UI as it stands, from Aa we can conclude that $(x)(Rx \rightarrow Tx)$—given an arbitrarily selected acute-angled shape, then all rectilinear shapes are triangles. The conclusion is evidently false, yet we can make the assumption true by simply selecting an acute-angled shape.

The fallacy involved here may be described by saying that we have no right to pass from the conclusion $Ra \rightarrow Ta$ to $(x)(Rx \rightarrow Tx)$, just because that conclusion rests on the *special assumption* concerning a that Aa. We have in fact proved that if our arbitrarily selected shape a is rectilinear then it is triangular, but only on the assumption that it is acute-angled as well. We can avoid this fallacy if, before we apply UI in passing from a proposition about a to a universal conclusion, we make sure that the assumptions on which the proposition about a rest do not include a special assumption concerning a itself; that is to say that, before we apply UI, we should make sure that 'a' does not appear in any of the assumptions on which the conclusion rests. This blocks successfully the fallacious move indicated above. For the conclusion $Ra \rightarrow Ta$ rested on the assumption Aa, in which a is mentioned, so that UI cannot be applied.

The applications of UI given earlier obey this restriction, as the reader should check for himself. For example, in the proof of 103 we applied UI to the conclusion Ga to obtain $(x)Gx$; but the assumptions on which Ga rested were $(x)(Fx \rightarrow Gx)$ and $(x)Fx$, in neither of which does 'a' appear. The restriction is easy to observe in practice: before applying UI to '$\ldots a \ldots$', in order to obtain '$(x)(\ldots x \ldots)$', we go through the assumptions on which '$\ldots a \ldots$' rests to ensure that 'a' nowhere appears in them.

The most direct form of the fallacy is observed in the following 'proof':

1	(1) Fa	A
1	(2) $(x)Fx$	1 UI

For example, taking 'F' as being odd, we may arbitrarily select, in the universe of numbers, an odd number, say 3, so that Fa becomes true; but it evidently does not follow that *all* numbers are odd, which is false. The move from (1) to (2) is prevented by the restriction, since (1) depends on *itself*, in which 'a' appears.

We have not, in fact, in this section given precise formulations of the rules UE and UI; this is delayed until Chapter 4, Section 1, where we present detailed formation rules for the predicate calculus analogous to those in Chapter 2, Section 1, for the propositional calculus. But the present intuitive account should enable the student to understand the elementary proofs given in the text and to work the exercises that follow. It is only in more sophisticated work that we require an exact statement of the quantifier rules.

EXERCISES

1 Translate the following arguments into the symbolism of the predicate calculus, and then show their validity by UE and propositional calculus rules:

(*a*) Jacques is a Frenchman; all Frenchmen are niggardly: therefore Jacques is niggardly. (' *m* ', ' *F* ', ' *N* ')

(*b*) Jacques is niggardly; no Frenchmen are niggardly; therefore Jacques is not a Frenchman. (' *m* ', ' *N* ', ' *F* ')

(*c*) William is not a Frenchman; only Frenchmen are avaricious; therefore William is not avaricious. (' *n* ', ' *F* ', ' *A* ')

(*d*) All male nurses are sympahetic; William is not sympathetic; William is male; therefore William is not a nurse. (' *M* ', ' *N* ', ' *S* ', ' *n* ')

(*e*) All Frenchmen except Parisians are kindly; Jacques is a Frenchman; Jacques is not kindly; therefore Jacques is a Parisian. (' *F* ', ' *P* ', ' *K* ', ' *m* ')

2 (i) Using UE and UI together with propositional calculus rules, show the validity of the following sequents:

(*a*) $(x)(Fx \rightarrow Gx),\ (x)(Gx \rightarrow -Hx) \vdash (x)(Fx \rightarrow -Hx)$

(*b*) $(x)(Fx \rightarrow -Gx),\ (x)(Hx \rightarrow Gx) \vdash (x)(Fx \rightarrow -Hx)$

(*c*) $(x)(Fx \rightarrow Gx),\ (x)(Hx \rightarrow -Gx) \vdash (x)(Fx \rightarrow -Hx)$

(*d*) $(x)(Gx \rightarrow -Fx),\ (x)(Hx \rightarrow Gx) \vdash (x)(Fx \rightarrow -Hx)$

(*e*) $(x)(Fx \rightarrow Gx) \vdash (x)\ Fx \rightarrow (x)\ Gx$

(*f*) $(x)(Fx \vee Gx \rightarrow Hx),\ (x)-Hx \vdash (x)-Fx$

(ii) For each of the following arguments, indicate which of the sequents (*a*)–(*d*) above exhibits its logical form (thus establishing the validity of the arguments):

(*a*) No Germans are Frenchmen; all Huns are German; therefore no Frenchmen are Huns.

(*b*) No Frenchmen are fanatics; all Huns are fanatics; therefore no Frenchmen are Huns.

(*c*) All Huns are fanatics; no Frenchmen are fanatics; therefore no Huns are Frenchmen.

(*d*) All Germans are fanatics; no fanatics are histrionic; therefore no Germans are histrionic.

3 THE EXISTENTIAL QUANTIFIER

As the universal quantifier is related to ' & ', so is the existential quantifier to ' v '. In the universe of 3 objects discussed in the last section, ' (x) *Fx* ' meant the same as ' *Fm* & *Fn* & *Fo* '. Now to say that there is *at least one x* with *F* in this universe is to say that *either m* has *F or n* has *F or o* has *F*. Hence here ' $(\exists x)Fx$ ' means the same as ' *Fm* v *Fn* v *Fo* '. In the case of an infinitely large universe, say that of the natural numbers, to say that there is a number with *F* or that some number has *F* is to say that either 0 has *F* or 1 has *F* or 2 has *F* or . . . . As we need the universal quantifier because we cannot write down an ' infinite conjunction ', so we need the existential quantifier because we cannot write down an ' infinite disjunction '.

Accordingly, the two rules for the existential quantifier can be seen as extensions of the rules vI and vE. Let us take the rule of existential quantifier introduction (EI) first. To establish a conclusion such as $(\exists x)Fx$, a natural premiss is something like *Fm*: given a particular object with *F*, we can conclude that *something* has *F*. Thus, in our universe of 3 objects, given any one of *Fm*, *Fn*, *Fo*, we can conclude $(\exists x)Fx$; or, in the infinite case, given any particular natural number with *F*, we can conclude that some number has *F*. If we bear in mind the disjunctive status of the existential quantifier, the analogy with vI should be obvious.

Hence the justification for EI is that, if a particular thing has a certain property, then something must have it, and EI will enable us to pass from premisses such as *Fm* and *Fn* to conclusion $(\exists x)Fx$, and from *Fm* & *Gm* and *Fn* & *Gn* to conclusion $(\exists x)(Fx$ & $Gx)$ (if *m* has both *F* and *G*, or if *n* has both *F* and *G*, then something has both *F* and *G*). Further, we extend the rule to apply also to premisses concerning arbitrarily selected objects *a*, *b*, *c*: for, if an arbitrarily selected thing has *F*, then again something has *F*. Hence, for example, EI will enable us to pass from premiss *Fa* v *Ga* to conclusion $(\exists x)(Fx$ v $Gx)$ (if an arbitrarily selected object has either *F* or *G*, then something has either *F* or *G*).

A very simple application of this rule occurs in the proof of the following (evidently valid) sequent:

104 $(x)Fx \vdash (\exists x)Fx$

1	(1)	$(x)Fx$	A
1	(2)	Fa	1 UE
1	(3)	$(\exists x)Fx$	2 EI

If everything has *F*, then in particular an arbitrarily selected object *a* has *F*, whence by EI something has *F*.

The rule of existential quantifier elimination (EE) can best be understood in the light of the rule vE. Given a disjunction A v B, it being desired to establish a conclusion C, we derive C first from A as assumption and then from B as assumption, knowing that, if C follows from *both* A *and* B, then, since one or the other holds, C must hold. Similarly, if we know in our 3-object universe that something has *F*, we know effectively

(1) *Fm* v *Fn* v *Fo*.

Seeking to establish a conclusion C, we might assume each disjunct of the complex disjunction in turn, knowing that if C follows from *all* those disjuncts, then, since one or other holds, C must hold. However, where an infinite universe is involved, $(\exists x)Fx$ is a kind of 'infinite disjunction', and there can be no question of deriving C from each of the infinitely many disjuncts. Now in the case of UI, we adopted the device of arbitrary names '*a*', '*b*', '*c*' just because we could not establish separately the infinitely many conjuncts that go to make the 'infinite conjunction' $(x)Fx$. For EE we may use the same device. Instead of showing that C follows from the separate assumptions *Fm*, *Fn*, *Fo*, we may show instead that C follows from the single assumption, *Fa*, that an arbitrarily selected object has *F*. The pattern of proof will then be: given $(\exists x)Fx$, and that C follows from assumption *Fa*, then C follows anyway. Here the proof of C from *Fa* is a condensed representation of possibly infinitely many derivations of C from all the disjuncts in the disguised disjunction $(\exists x)Fx$. We may call *Fa* here, I hope suggestively, the *typical disjunct* corresponding to the existential proposition $(\exists x)Fx$.

Thus the justification for EE is somewhat as follows. If something has a certain property, and if it can be shown that a conclusion C follows from the assumption that an arbitrarily selected object has that property, then we know that C holds; for if something has the property, and no matter which has it then C holds, then C holds anyway. The conclusion C will of course, as in vE, rest on any assumptions on which the existential proposition rests, and on any assumptions used to derive C from the corresponding typical disjunct apart from the disjunct itself. And on the right-hand side we shall cite three lines: (i) the line where the existential proposition occurs; (ii) the line where the typical disjunct is assumed; and (iii) the line where C is drawn as conclusion from the typical disjunct as assumption.

These new rules are illustrated by the following proofs:

105 $(x)(Fx \rightarrow Gx), (\exists x)Fx \vdash (\exists x)Gx$

1	(1) $(x)(Fx \rightarrow Gx)$	A
2	(2) $(\exists x)Fx$	A
3	(3) Fa	A
1	(4) $Fa \rightarrow Ga$	1 UE
1,3	(5) Ga	3,4 MPP
1,3	(6) $(\exists x)Gx$	5 EI
1,2	(7) $(\exists x)Gx$	2,3,6 EE

Given that everything with F has G and that something has F, we show that something has G. We assume, preparatory to EE, that an arbitrarily selected object a has F at line (3), and then conclude (line (6)) that something has G. We are now ready for a step of EE; given an existential proposition to the effect that something has F at line (2) and a derivation of the desired conclusion from the corresponding typical disjunct at line (6), we obtain the conclusion again at line (7). We cite on the right line (2), the existential proposition, line (3), the typical disjunct, and line (6), the conclusion obtained from that assumption. The conclusion now rests upon whatever assumptions the existential proposition rests upon—here merely itself—and any assumptions used to obtain the conclusion from the typical disjunct Fa apart from Fa itself, which gives just (1) and (2).

The analogy with vE can be brought out by supposing that, as a special case, we are dealing with a 2-object universe, containing just m and n. Then, for this universe, $(\exists x)Fx$ amounts to $Fm \vee Fn$, and $(\exists x)Gx$ to $Gm \vee Gn$. The corresponding proof with vE in place of EE would go as follows:

			1	(1) $(x)(Fx \rightarrow Gx)$	A
			2	(2) $Fm \vee Fn$	A
3	(3) Fm	A	3′	(3′) Fn	A
1	(4) $Fm \rightarrow Gm$	1 UE	1	(4′) $Fn \rightarrow Gn$	1 UE
1,3	(5) Gm	3, 4 MPP	1,3′	(5′) Gn	3′, 4′ MPP
1,3	(6) $Gm \vee Gn$	5 vI	1,3′	(6′) $Gm \vee Gn$	5′ vI
			1,2	(7) $Gm \vee Gn$	2,3,6,3′,6′ vE

Here the lines (3′)–(5′) exactly mirror (3)–(5) with '*n*' in place of '*m*'. The lines (3)–(6) of our actual proof of 105 condense these twin arguments into one argument, by the employment of arbitrary names in place of the proper names '*m*' and '*n*', and by using the typical disjunct '*Fa*' in place of the separate disjuncts '*Fm*' and '*Fn*'.

106 $(x)(Gx \rightarrow Hx), (\exists x)(Fx \,\&\, Gx) \vdash (\exists x)(Fx \,\&\, Hx)$

1	(1) $(x)(Gx \rightarrow Hx)$	A
2	(2) $(\exists x)(Fx \,\&\, Gx)$	A
3	(3) $Fa \,\&\, Ga$	A
1	(4) $Ga \rightarrow Ha$	1 UE
3	(5) Ga	3 &E
1,3	(6) Ha	4,5 MPP
3	(7) Fa	3 &E
1,3	(8) $Fa \,\&\, Ha$	6,7 &I
1,3	(9) $(\exists x)(Fx \,\&\, Hx)$	8 EI
1,2	(10) $(\exists x)(Fx \,\&\, Hx)$	2,3,9 EE

The strategy here should be clear. To prove $(\exists x)(Fx \,\&\, Hx)$ from $(\exists x)(Fx \,\&\, Gx)$, we aim for the same conclusion from $Fa \,\&\, Ga$, the corresponding typical disjunct. Since everything with *G* has *H*, from *Ga* we can infer *Ha*, hence *a* has both *F* and *H*, and so something has both *F* and *H*. The conclusion at (10) rests on (2), the original existential proposition, and (1), which was used to obtain the conclusion from (3), as we see at line (9).

These two proofs illustrate a general tip for proof-discovery. Given $(\exists x)(\ldots x \ldots)$ and desiring to prove a conclusion C, you should assume (. . . *a* . . .) as typical disjunct and try to obtain C from it. For, if you succeed, EE will give you just this conclusion. Once (. . . *a* . . .) has been assumed, reasoning of the propositional calculus type will generally assist in the derivation of C.

As in the case of UI, the use of arbitrary names with EE necessitates certain restrictions if fallacies are to be avoided. In the case of UI, we required that the arbitrary name in question should not appear in the assumptions on which the conclusion drawn rested. For EE we require that the arbitrary name in question shall not appear either in the conclusion C drawn or in the assumptions used to derive C from the typical disjunct (though of course it will appear in the typical disjunct itself).

To see that the arbitrary name must not appear in the conclusion C, we need only observe that otherwise we could prove, given that something has *F*, that everything has *F*.

1	(1) $(\exists x)Fx$	A
2	(2) Fa	A
1	(3) Fa	1,2,2 EE
1	(4) $(x)Fx$	3 UI

The step of UI is correct, since 1 does not contain '*a*'. But the step of EE is incorrect because the conclusion in question, here *Fa*, does contain '*a*'. It does not follow from something's having *F* that an arbitrarily selected object has *F*, though of course *Fa* follows from itself. To see that the arbitrary name must not appear in the assumptions (apart from the typical disjunct) used to obtain C, consider the following 'proof':

1	(1) Fa	A
2	(2) $(\exists x)Gx$	A
3	(3) Ga	A
1,3	(4) Fa & Ga	1,3 &I
1,3	(5) $(\exists x)(Fx$ & $Gx)$	4 EI
1,2	(6) $(\exists x)(Fx$ & $Gx)$	2,3,5 EE

The conclusion, that something has both *F* and *G*, is here reached from the two assumptions that an arbitrarily selected object has *F* and that something has *G*. Now, let *F* be being even, and *G* be being odd: then I can select a number a which is even, so that Fa becomes true, and there are odd numbers, so that $(\exists x)Gx$ is also true. But it is false that any number is both odd and even. The step of EE is unsound, because the conclusion at line (5) rests on (1) which contains 'a'.

The new restriction is again easy to observe in practice. For example, to see that the step of EE at line (10) of 106 is correct, we inspect line (9); the conclusion there does not contain 'a': of the two assumptions on which it rests, (3), the typical disjunct, of course contains 'a' but (1) does not; thus the restriction is met.

Since arbitrarily selected objects play a large part in our work, it may be as well to attempt to clarify their position. Let F be some property, and a an arbitrarily selected object from some universe; then, given that everything has F, a has F, but not conversely. We accept as valid the sequent $(x)Fx \vdash Fa$, but not the sequent $Fa \vdash (x)Fx$; and we reject the latter because a, though arbitrarily selected, may not be *typical*. On the other hand, by UI, under certain conditions we pass from premiss $(\ldots a \ldots)$ to conclusion $(x)(\ldots x \ldots)$; however, the conditions involved are such as to ensure that a *is* here typical, for we stipulate that $(\ldots a \ldots)$ shall not rest on any special assumptions about a. We also declare that, given that a has F, something has F, but not conversely. We accept as valid the sequent $Fa \vdash (\exists x)Fx$ but not the sequent $(\exists x)Fx \vdash Fa$, and we reject the latter because a, being arbitrarily selected, may not be one of the given objects with F. On the other hand, by EE, under certain conditions we can derive conclusions obtained from Fa directly from $(\exists x)Fx$, as though what Fa implied $(\exists x)Fx$ implied also; however, the conditions involved are such as to ensure that any such conclusion is obtained from Fa only on the understanding that a is typical—no special assumptions about a other than Fa are made and the conclusion does not concern a—and so can be taken as one of the given objects with F. Thus the claim that an arbitrarily selected object has F must be distinguished both from the claim $(x)Fx$ and the claim $(\exists x)Fx$, though it is derivable from the former and the latter is derivable from it.

EXERCISES

1 Using quantifier and propositional calculus rules, show the validity of the following sequents: